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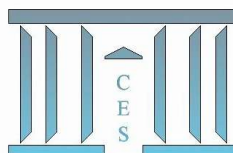
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Under Imperfect Competition**

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# Markovian Equilibrium in a Model of Investment Under Imperfect Competition

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## **Abstract**

This paper develops and analyzes a dynamic model of partially irreversible investment under cournot competition and stochastic evolution of demand. In this framework, I characterize the markov perfect equilibrium in which player's strategies are continuous in the state variable. There exists a zone in the space of capacities, named the no-move zone, such that if firms capacity belongs to this area, no firm invest nor disinvest at the equilibrium. Thereby, initial asymmetry between firms capacity can be preserved. If firms are outside this area, they invest in order to reached the no-move zone. The equilibrium as an efficiency property: the point of this area which is reached by the firms minimizes the investment cost of the all industry.

**Keywords:** Capacity investment and disinvestment. Dynamic stochastic games. Markov perfect equilibrium. Real option games.

**JEL classification:** D43 L13 L25

## **Abstract**

Cet article étudie un modèle dynamique d'investissement partiellement irréversible en concurrence à la Cournot lorsque la demande évolue aléatoirement. Dans ce cadre, je caractérise l'équilibre markovien pour lequel les stratégies des joueurs sont continues dans la variable d'état. Il existe une zone dans l'espace des capacités, appelé No-move zone, tel que, à l'équilibre, si le vecteur des capacités appartient à cette zone, les entreprises gardent leurs capacités constante. Ainsi, une asymétrie initiale entre les tailles des entreprises peut être préservée. Lorsque les entreprises sont en dehors de cette zone, elles investissent (ou désinvestissent) afin de rejoindre la No-move zone. L'équilibre possède une propriété de l'efficacité: le point de cette zone qui est atteint par les entreprises est le point qui minimise le coût d'investissement de l'ensemble de l'industrie.

**Mots-clefs:** Investissement en capacité. Jeux différentiel. Equilibre markovien. Option réelle.

# 1 Introduction

Capacity expansion or reduction under uncertainty is one of the most important decisions that firms can make. It impacts their immediate profit and creates long-run commitment. In a dynamic setting, the investment pattern by a monopoly is well known. Because of uncertainty, the firm has incentives to delay a profitable project (in expectation) in order to wait for more information about the demand. This is the theory of real options. What happens in imperfect competition becomes the theory of real option games. (For recent surveys, see for instance [Boyer, Gravel and Lasserre \(2004\)](#), [Azevedo and Paxson \(2014\)](#), or [Chevalier-Roignant et al \(2011\)](#)).

In this literature several authors focused on capacity decision under uncertainty. In these models, at each time the profit made by a firm depends on its size (i.e. its capacity), the size of its opponents, and a parameter which evolves randomly over time (which can be a parameter of demand or cost, the important point being that future evolutions are unknown). As this parameter evolves, firms wish to adapt their sizes. Firms can either invest to increase their capacity, disinvest to reduce it, or let their capacity depreciate at its natural rate. Investment is said perfectly reversible when the cost of investing is equal to the cost of disinvesting, in which case firms can perfectly adapt to the stochastic evolution of the parameter. Such repeated game framework is classic in industrial organization. However, in reality, increasing the size implies hiring new employees, building new factories or office, buying new equipment, and so on... These investments are usually at least partially sunk, so the firm's size decisions are not perfectly reversible. Investment is said totally irreversible when firms cannot reduce their sizes, and investment is partially irreversible when firms can decrease their size by disinvesting (but with a scrap value inferior to the cost of investing) or by depreciation. In these cases, the theory of real option game links the hysteresis due to the irreversibility of investment and the imperfect competition.

This paper studies capacity accumulation when competition is imperfect and investment partially irreversible. It shows that the irreversibility of investment implies a preservation of the asymmetry in firms capacity: if a firm owns more capacity than its opponent, the bigger firm still have a capacity more important than its opponent at the equilibrium. The equilibrium has also an efficient capacity: the state of the industry at the equilibrium min-

minizes the cost of investment of the all industry. This features are consistent with dynamics and uncertainty.

Several authors study capacity's choices under imperfect competition and uncertain demand. [Abel et al \(1996\)](#) and [Merhi and Zervos \(2007\)](#) focus on the monopoly behavior under partially irreversible investment. They exhibit two boundaries, such that the firm keep its capacity constant when the demand is between the two boundaries, invest when the demand is higher than the left-hand boundary, and disinvest otherwise. This creates an hysteresis on the investment decision. When competition is imperfect, [Boyer et al \(2004\)](#) and [Boyer, Lasserre and Moreaux \(2012\)](#) assume irreversible investment and lumpy investment. There exists some levels of demand where a firm has no interest too buy a new unit of capacity if its competitors also install a new unit, but has an interest to do so if its competitor does not add a capacity. This creates different kinds of equilibria, some in which firms want to preempt their opponents, and some where they behaves less aggressively. When capacity is smooth and investment irreversible, [Grenadier \(2002\)](#), [Back and Paulsen \(2009\)](#) and [Chevalier-Roignant, Huchzermeier and Trigeorgis \(2011\)](#) focus on the choice of the duopoly. [Grenadier \(2002\)](#) assume that the strategy of a firm is to invest until a boundary, which depends on the level of demand, and describes the Nash equilibrium with such strategies. [Back and Paulsen \(2009\)](#) show that this equilibrium is open-loop, i.e. it is the Nash equilibrium when the strategy of a firm is its path of capacity and firms can commit to their strategy. Usually, Open-loop equilibria are not sub-game perfect, and [Chevalier-Roignant, Huchzermeier and Trigeorgis \(2011\)](#) focus on markovian equilibrium. The authors describe the optimal markovian best response of the firms. However, the linearity of the investment cost implies infinite value for the amount of investment (the capacity jumps, as there is no interest to delay purchases or sells of capital), which prevents the characterization of the markovian equilibrium. In order to fill this gap, I introduce a possibility of jumps in the formulation of the strategies and characterizes the markovian equilibrium, in case partially irreversible investment.

To do so, I first study the simplest investment game. In a one shot model, firms have initial capacities and can invest or disinvest in a partially irreversible way (firms can buy or sell capacities at linear but different prices). I exhibit an area in the space of capacities, named the no-move zone. If the capacities of the firms are inside this no-move zone, no firm will neither invest nor disinvest. So any point of this no-move zone is a possible equilibrium,

given some initial capacities. For some given initial capacities, the equilibrium is the point of the no-move zone which minimizes the costs of investment and disinvestment for the all industry. This efficiency result holds even though firms have a priori no interest in coordinating their decisions. Furthermore, as long as the prices of investment and disinvestment are not equal, the no-move zone is not reduced to one point, and an initial asymmetry in capacities can be preserved.

In continuous time, the linearity of investment cost gives an incentive to invest as soon as possible. This creates a difficulty of modeling the strategies of the firms, as the desired flow of investment is infinite. In that context, I introduce a possibility of jump for the firm and solve the markovian equilibrium of this game. There exists a no-move zone. At each time firms reach the point of the no-move zone which minimizes the industry costs of investment and disinvestment. In this sense, they behave as in the one shot game. This result is valid when demand evolves in a deterministic or a stochastic way.

The outline of this paper is straightforward. Section 2 studies the one-shot model. Section 3 presents the dynamic model and characterizes the markovian equilibria. Section 4 concludes. All omitted proofs are reported in appendix.

## 2 Investment in the one-period game

### 2.1 The one-shot game model

The aim of this subsection is to abstract from dynamics and uncertainty issues, in order to focus on the effect of partial irreversibility of investment. To do so, I present a simple static model of competition in capacity.

More precisely, consider a market with  $n$  firms competing *à la Cournot* in capacities. Each firm  $i$  starts with some amount of capital  $k^i$ , which can be extended or reduced through buying or selling some assets. Purchases are made at a linear price  $p^+$ , and sales at a (also linear) price  $p^-$  (with  $p^- \leq p^+$ ). I call  $K^i$  the capacity finally installed by firm  $i$ . Let  $k$  be the vector of industry's initial capacities and  $K$  the vector of installed capacities. For firm

$i$ , the cost of installing a new capacity is:

$$C(K^i, k^i) = \begin{cases} p^+ (K^i - k^i) & \text{if } K^i \geq k^i \\ p^- (K^i - k^i) & \text{if } K^i < k^i \end{cases}. \quad (1)$$

Firms produce and sell an homogenous good, at a price depending of the total quantity  $\bar{q} = \sum_{i=1}^n q^i$ . Each firm's production depends of its capacity, according to the technology  $q^i = K^i$ ,<sup>1</sup> and has a cost,  $c_i(q^i)$ . Such technology is classic in dynamic investment models, and has been used by [Fudenberg and Tirole \(1983\)](#), [Grenadier \(2002\)](#), [Merhi and Zervos \(2007\)](#) among others<sup>2</sup>. So, by selling the quantity  $K^i$ , firm  $i$  obtains a payoff of:

$$\pi^i(K) = P(\bar{K}) K^i - c_i(K^i), \quad (2)$$

where  $\bar{K} = \sum_{j=1}^n K^j$ . The profit function of firm  $i$  is thus:

$$\Pi^i(K, k) = \pi^i(K) - C(K^i, k^i). \quad (3)$$

Note that if  $p^- = p^+$ , the investment decision is totally reversible, and the initial capacity has no impact. The smaller is  $p^-$ , the more irreversible are the capacities of the firm, and, at the limit, when  $p^- = -\infty$ , investment is totally irreversible, as in [Grenadier \(2002\)](#), [Back and Paulsen \(2009\)](#), [Boyer, Lasserre and Moreaux \(2012\)](#), and others.

In order to ensure the existence of the equilibrium I make the following hypothesis:

**H1:** For each  $i = 1, \dots, n$ ,  $c_i(\cdot)$  is a twice-differentiable positive function such that  $c'_i \geq 0$ ,  $c''_i \geq 0$ .  $P(\cdot)$  is also a twice-differentiable positive function, with  $P' < 0$ ,  $P'' < 0$  when  $P$  is

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<sup>1</sup>In the one-shot model, this technology can be seen as the result of an endogenized game, in which firms buy capacities and then play a Cournot competition limited by the capacity previously bought. Indeed, firms have no interest to invest in capacity which will not be used to produce, as its opponents only react to the final quantity. (Except if the disinvestment price is negative. In this case a firm could wish to keep its unused capacity in order to avoid a disinvestment case. For example, this is sometimes the case for polluted production sites, for which the cost of decontamination is more important than the cost of conservation of this asset.)

<sup>2</sup>As it was shown in [Reynolds \(1987\)](#), this technology assumption is also the result of a dynamic games with limited Cournot competition, without uncertainty. However, when there is uncertainty, firms have an incentive to keep their unused capacity for a possible further use, when demand increases. Assuming that quantities are equal to capacities permits to avoid such adaptability effects and focus on the direct effect of uncertainty on capacity choice.



strictly positive.<sup>3</sup> Furthermore, for all  $i = 1, \dots, n$  and  $q \in \mathbb{R}_+^n$ ,  $P''(\bar{q})q^i < -c_i''(q^i)$ .

## 2.2 Best responses

This part describes the best response of firm  $i$ . Assume that for all  $j \neq i$ , firm  $j$  installs a capacity  $K^j$ . Obviously, the marginal revenue of capacity for firm  $i$  depends of the choices of its opponents, and I note  $\frac{\partial \pi^i}{\partial K^i}^{-1}(x)$  the inverse function of the marginal revenue of capacity, so

$$\frac{\partial \pi^i}{\partial K^i}^{-1}(x) = \hat{K} \Leftrightarrow \frac{\partial \pi^i}{\partial K^i}(K^1, \dots, K^i = \hat{K}, \dots, K^n) = x.$$

By concavity of  $\pi$  in  $K^i$ , such inverse function is well defined and increasing. Then, firm  $i$  has three possible choices:

- to invest ( $K^i > k^i$ ), so making a profit:

$$\Pi^i = \pi^i(K) - p^+(K^i - k^i),$$

leading to an optimal choice of capacity  $\frac{\partial \pi^i}{\partial K^i}^{-1}(p_k^+)$ . Hence, if  $k^i > \frac{\partial \pi^i}{\partial K^i}^{-1}(p_k^+)$ , increasing its capacity decreases firm's profit, and the firm has no interest to invest.

- to disinvest ( $K^i < k^i$ ), so making a profit:

$$\Pi^i = \pi^i(K) + p^-(k^i - K^i),$$

leading to an optimal capacity  $\frac{\partial \pi^i}{\partial K^i}^{-1}(p^-)$ . Thus, the firm has no interest in disinvesting if  $k^i < \frac{\partial \pi^i}{\partial K^i}^{-1}(p^-)$  (which is higher than  $\frac{\partial \pi^i}{\partial K^i}^{-1}(p^+)$ ).

◦ the last possibility is to do nothing (the firm neither invest nor disinvest). Indeed, as  $\frac{\partial \pi^i}{\partial K^i}^{-1}(\cdot)$  is increasing, the initial capacity of the firm can be greater than  $\frac{\partial \pi^i}{\partial K^i}^{-1}(p_k^+)$ , so the firm has no interest to invest more, but also smaller than  $\frac{\partial \pi^i}{\partial K^i}^{-1}(p^-)$ , so the firm has also no interest to disinvest.

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<sup>3</sup>Theorem 1 (presented page 9) holds under less restrictive conditions. Indeed, our proof rests on the third theorem of [Novshek \(1985\)](#), and the linearity of cost of investment and disinvestment. However, as the point of interest is the dynamic game, and as some regularity is needed for the existence of the dynamic differentiable equations, I place ourselves in the assumption made by [Szidarovszky and Yakowitz \(1977\)](#).

Therefore there exists two thresholds,  $\frac{\partial \pi^i}{\partial K^i}^{-1}(p^-)$  and  $\frac{\partial \pi^i}{\partial K^i}^{-1}(p^+)$  such that the firm does not wish to invest nor disinvest if its capital is between this thresholds, invest until  $\frac{\partial \pi^i}{\partial K^i}^{-1}(p^+)$  if its initial capacity is small, and disinvest until  $\frac{\partial \pi^i}{\partial K^i}^{-1}(p^-)$  if its initial capacity is large. This can be summarized in the following proposition:

**Proposition 1:** The best response of firm  $i$  is:

$$K_{BR}^i = \left\{ \begin{array}{ll} \frac{\partial \pi^i}{\partial K^i}^{-1}(p^+) & \text{if } k^i < \frac{\partial \pi^i}{\partial K^i}^{-1}(p^+) \\ k^i & \text{if } k^i \in \left[ \frac{\partial \pi^i}{\partial K^i}^{-1}(p^+), \frac{\partial \pi^i}{\partial K^i}^{-1}(p^-) \right] \\ \frac{\partial \pi^i}{\partial K^i}^{-1}(p^-) & \text{if } k^i > \frac{\partial \pi^i}{\partial K^i}^{-1}(p^-) \end{array} \right\}. \quad (4)$$

■

Of course, this best response depends of the capacities of other firms, as  $\frac{\partial \pi^i}{\partial K^i}$  depends of the capacity of all firms. Graphic 1 represents the best response of firm 2 in the space of capacity, for a duopoly with linear demand and no production costs.

[Insert G1]

In this graphic, we can see the existence of an area in the space of capacities,  $\Gamma^2$ , bounded by  $\frac{\partial \pi^2}{\partial K^2}^{-1}(p^-)$  and  $\frac{\partial \pi^2}{\partial K^2}^{-1}(p^+)$ , such that, it is never optimal for firm 2 to be outside this area. Thus, if there is an equilibrium, it belongs to  $\Gamma^i$  for all firm  $i$ , so it belongs to the intersection of these areas. Let  $H$  be this intersection. We know that all equilibria belong to  $H$ . Furthermore, assume that the initial distribution of capacity belongs to  $H$ . Then, if all players except  $i$  keep their capacity constant, then the best response of  $i$  is to keep its initial capacity. In the case of duopoly with linear demand and no production cost, this can be seen in graphic 2. The area  $H$  is called the no-move zone.

[Insert G2]

## 2.3 Characterization of the equilibrium

In this subsection I show the existence of an area in the set of space capacities, the no-move zone  $H$ , such that all equilibria belong to  $H$ . This no-move zone is defined by,

$$H = \left\{ K \in \mathbb{R}_+^n : \forall i \in \{1, \dots, n\}, \frac{\partial \pi^i}{\partial K^i}(K) \in [p^-, p^+] \right\}, \quad (5)$$

which can be rewritten as

$$H = \left\{ K \in \mathbb{R}_+^n : \forall i \in \{1, \dots, n\}, P(\bar{K}) + P'(\bar{K}) K^i - c_i(K^i) \in [p^-, p^+] \right\}. \quad (6)$$

If the initial capacities of the firms are in  $H$ , the equilibrium is to keep the same capacities.

If the initial capacities do not belong to  $H$ , the equilibrium is described by theorem 1.

**Theorem 1:** Assume  $H1$ . Then, there exists only one Nash equilibrium  $K^*$ , which satisfies:

$$K^* = \arg \min_{K \in H} \sum_{i=1}^n C(K^i, k^i). \quad (7)$$

This condition is equivalent to the distance condition,

$$K^* = \arg \min_{K \in H} \sum_{i=1}^n |K^i - k^i|. \quad (8)$$

■

Theorem 1 provides existence and uniqueness for the equilibrium, and its characterization. To understand this characterization, let focus on the no-move zone. When investment is totally reversible (when  $p^+ = p^-$ ) the no-move zone is reduced to a unique point, and no industry efficiency appears. This is the usual Cournot competition<sup>4</sup>. When investment is not totally reversible, the no-move zone is a set, not reduced to a singleton. In this area the marginal revenue of each firm is inferior to the price of adding a new capacity, but superior to the price of selling some capacity, so no firms wish to invest nor disinvest. The size of the no-move zone is thus an increasing function of the irreversibility of investment.<sup>5</sup> By theorem

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<sup>4</sup>With a cost  $c_i(K^i) + p^+ K^i$ .

<sup>5</sup>More precisely, when  $p^+$  increases and  $p^-$  decreases, the size of no-move zone increases according to (5). When both  $p^+$  and  $p^-$  increases (or decreases), the evolution of the size of the no-move zone is unknown even if the difference  $p^+ - p^-$  increases, as it depends of the marginal revenue of each firm.

1,  $H$  is also the set of all possible equilibria, in the sense that all equilibrium belongs to  $H$  and all point of  $H$  can be an equilibrium for some initial value.

Inside the no-move zone, there is a property of no-reaction to competition. Indeed, assume that the firms decide to install a vector of capacity which belongs to  $H$ . Then, if a firm changes its level of capacity (and if the vector of capacity still belongs to  $H$ ), its opponents will not invest nor disinvest in reaction to this change of capacity. This is due to the fact that if the evolution of the firm capacity has an impact on its opponent's revenue, this impact is not sufficient to generate a change in capacity due to the irreversibility of investment. Inside the no-move zone, each firm is thus myopic, and maximizes its profit assuming that the other firms have constant capacities. Theorem 1 shows that this myopicity leads to a market efficiency. Indeed, the equilibrium is the point of the no-move zone which minimizes the industry cost of investment.

Graphic 3 presents which points of the no-move zone will be an equilibrium, in function of the initial capacities, for a duopoly with linear demand and no production costs. As it can be seen on the graphic, firms with different initial capacities can still be asymmetric at the equilibrium, and there are different possible symmetric equilibria, even when firms have the same profit function.

*[Insert G3]*

### 3 Investment in a dynamic game

#### 3.1 The dynamic model

This section presents a dynamic model of investment in capacity under imperfect competition. It shows that, under some smoothness condition, the markovian equilibria are similar to the one shot game previously considered. There exists a no-move zone and the firms join it as soon as possible minimizing the investment cost of the all industry. This permits to characterize the equilibria.

Let  $k_t^i$  be the capital of firm  $i$  at time  $t$ , time is continuous and capacity partially reversible. Let  $\pi^i(A_t, k_t)$  be the instantaneous payoff of firm  $i$ ,  $k_t$  being the vector of firms capacities and  $A_t$  the parameter of uncertainty, following a diffusion process:

$$dA_t = \beta(A_t)dt + \sigma(A_t)dW_t, \quad (9)$$

where  $W_t$  is a standard Wiener process. I assume Cournot competition. Let  $P_{A_t}(\cdot)$  be the inverse demand function, depending of the level of demand  $A_t$ , and let  $c^i(\cdot)$  be the production cost of firm  $i$ , then the instantaneous revenue of firm  $i$  is

$$\pi^i(A_t, k_t) = P_{A_t}(\bar{k}_t) k_t^i - c^i(k_t^i), \quad (10)$$

where  $\bar{k}_t = \sum_{i=1}^n k_t^i$ , as previously. The interest rate is  $r$  and the purchase price of capacity  $p^+$ , whereas the selling price is  $p^-$ . For each stochastic process  $k_t^i$ , let  $k_t^{i+}$  and  $k_t^{i-}$  be the respectively increasing and decreasing processes such that<sup>6</sup>:

$$k_t^i = k_t^{i+} + k_t^{i-}. \quad (11)$$

The total expected profit of firm  $i$  at time 0 is thus:

$$\Pi^i = E \left[ \int_0^{+\infty} e^{-rt} \pi^i(A_t, k_t) dt - p^+ \int_0^{+\infty} e^{-rt} dk_t^{i+} + p^- \int_0^{+\infty} e^{-rt} dk_t^{i-} \mid A_0 \right]. \quad (12)$$

The objective of firm  $i$  is to choose the process  $k_t^i$  which maximizes its own expected profit, given the initial levels of capital and demand. Obviously, the optimal process will depend of the processes chosen by the other players. To properly define the game, I must define the strategic variable and the equilibrium concept used by the players.

I focus on markov perfect equilibria. The strategy of a player is a function of a state variable, representing the level of the demand and firms capacity at time  $t$ , which determines the futur evolution of the player capacities. Markov perfect equilibria have the advantage to be sub-game perfect, and to avoid time inconsistency.

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<sup>6</sup>This definition is valid for all left-continuous  $A_t$ -adapted stochastic process with finite variation. The assumption of finite variation is natural with the investment cost considered. Indeed, a firm with an infinite variation of its capacity will pay an infinite cost of investment and disinvestment (as  $p^+ > p^-$ ). However, its future revenue is finite (due to hypothesis  $H4$ ), and thus a strategy with infinite variation would lead to a negative and infinite profit, which is obviously not optimal.

In our framework, there is a difficulty to define the strategic variable. Indeed, the linearity of investment cost might create an incentive to install the desired capacity as soon as possible. This creates a jump in the capacity process of the firms. However, the usual theory of differential games does not allow the capacity processes to jump (see also [Back and Paulsen \(2009\)](#) for the same observation on the definition of strategies). Indeed, in differential games, the evolution of the state variable is usually determined by  $I_t^i$ , the investment done by firm  $i$  at date  $t$ , so that the capital of each firm is determined by the following differential equation:

$$\frac{\partial k_t^i}{\partial t} = I_t^i. \quad (13)$$

This definition implicitly assumes that the capacity is a continuous function of the time. A markovian strategy for firm  $i$  being a function of the state variable (demand level and capacities),  $I_t^i = \tilde{I}^i(A_t, k_t)$ . As it can be seen in [Chevalier-Roignant, Huchzermeier and Trigeorgis \(2011\)](#), the Bellman formula gives in that case:

$$r\Pi^i(A, k) = \sup_{I^i} \left\{ P_A(\bar{k}) k^i - p^+(I^i)_+ - p^-(I^i)_- + I^i \frac{\partial \Pi^i}{\partial K^i} + \beta(A) \frac{\partial \Pi^i}{\partial A} + \frac{\sigma^2(A)}{2} \frac{\partial^2 \Pi^i}{(\partial A)^2} + \sum_{j=1, j \neq i}^n \tilde{I}^j(A, k) \frac{\partial \Pi^i}{\partial K^j} + \sum_{j=1, j \neq i}^n \sum_{h=1, h \neq i}^n \tilde{I}^h(A, k) \tilde{I}^j(A, k) \frac{\partial^2 \Pi^i}{\partial K^j \partial K^h} \right\}. \quad (14)$$

The optimal investment policy maximizes  $\frac{\partial \Pi^i}{\partial K^i} I^i - p^+(I^i)_+ - p^-(I^i)_-$ . So, if  $\frac{\partial \Pi^i}{\partial K^i} \in [p^-, p^+]$ , the firm  $i$  has no interest to invest nor to disinvest. Otherwise, the optimal flow of investment  $I_t^i$  is infinite: the firm installs its optimal capital (capital in the region  $\frac{\partial \Pi^i}{\partial K^i}^{-1}([p^-, p^+])$ ) instantly. The optimal capital policy of the firm is thus to jump in the area  $\frac{\partial \Pi^i}{\partial K^i}^{-1}([p^-, p^+])$ , and to do nothing as long as the capital stays in this area. So the optimal strategy cannot be defined by the investment variable, as the linearity of the cost of investment implies non-continuous capital strategies.

To address this difficulty, I introduce a new control variable,  $K_t^i$ , which is the capacity desired by firm  $i$  at time  $t$ . If this desired capacity is equal to the installed capacity,  $k_t^i$ , the firm continue to invest in a continuous way. If the desired capacity is different from the installed capacity, the firm installs the desired capacity. Formally, this is expressed in the following definition.

**Definition:** The investment game previously considered is in its markovian state-control form if:

- (i) a strategy of player  $i$  is a pair  $(K_t^i, I_t^i)_{t \in \mathbb{R}_+}$ , where for each  $t$ ,  $(K_t^i, I_t^i) \in \mathbb{R}_+^2$ .
- (ii) the state variable at time  $t$ ,  $k_t = (k_t^1, \dots, k_t^n)$ , is defined by the two equations:

$$\frac{\partial^+ k_t^i}{\partial t} = I_t^i, \quad (15)$$

and

$$\lim_{\substack{s \rightarrow t \\ s > t}} k_s^i = K_t^i, \quad (16)$$

where  $k_0$  is the given initial level of capital, and  $\frac{\partial^+}{\partial t}$  denotes the right-hand derivatives.<sup>7</sup>

- (iii) the strategy of player  $i$  is markovian if its strategy is only function of the state variable (firms capacity and level of demand), i.e. if there exists  $\tilde{K}^i(., .)$  and  $\tilde{I}^i(., .)$  such that

$$K_t^i = \tilde{K}^i(A_t, k_t) \text{ and } I_t^i = \tilde{I}^i(A_t, k_t). \quad \square \quad (17)$$

Equation (16) states that the installed capacity  $k_t^i$  is continuous if and only if  $K_t^i = k_t^i$ . When the strategies are markovian,  $K_t^i$  is a function of the state variable, and firm  $i$  jumps as long as  $k_t^i \neq \tilde{K}^i(A_t, .)$ . Graphic 4 presents the evolution of the capacity of firm  $i$  when player's  $i$  strategy is

$$\tilde{I}^i(A_t, k_t) = 1, \quad \tilde{K}^i(A_t, k_t) = 1_{\{k_t^i < 2\}} k_t^i + 1_{\{k_t^i \geq 2\}},$$

and  $A_t$  and  $(k_t^j)_{j \neq i}$  are constant. As long as the installed capacity,  $k_t^i$ , is inferior to two, the firm invests continuously, and when installed capacity reaches two, the desired capacity,  $K_t^i$  differs from the installed capacity, and the firm disinvests instantly a unit of capacity.

[Insert G4]

The definition of the markovian state-control form extends the possibility of the stochastic processes of capacity from the class of continuous processes implicitly assumed by (13) to the class of the left-hand continuous stochastic processes with right-hand derivative. However, this definition allow multiple jumps: a jump of firm  $i$  can imply a jump of its

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<sup>7</sup>As process  $k_t$  can be discontinuous in  $t$ , the right-hand derivative is defined by:  $\frac{\partial^+ k_t^i}{\partial t} = \lim_{n \rightarrow +\infty} \frac{l - k_{t+h}^i}{h}$ , where  $l = \lim_{\substack{s \rightarrow t \\ s > t}} k_s^i$ .

opponents in reaction, which can bring a new jump of firm  $i$ , and so on... In order to prevent such multiple jumps, I introduce the following assumption.

**C1:** The strategies of the firms  $i$  verifies assumption C1 if, for all  $(A, k) \in \mathbb{R} \times \mathbb{R}_+^n$ ,

$$\tilde{K} \left( A, \tilde{K}(A, k) \right) = \tilde{K}(A, k). \quad (18)$$

A markov perfect equilibrium in the markovian state-control form is defined as a vector of functions  $\left( \tilde{K}^* (.,.), \tilde{I}^* (.,.) \right) = \left( \tilde{K}^{*1} (.,.), \tilde{I}^{*1} (.,.), \dots, \tilde{K}^{*n} (.,.), \tilde{I}^{*n} (.,.) \right)$  such that, for all firm  $i \in \{1, \dots, n\}$  and for all other markovian strategy of firm  $i$ ,  $\left( \tilde{K}^i (.,.), \tilde{I}^i (.,.) \right)$ ,

$$\forall (A, k) \in \mathbb{R} \times \mathbb{R}_+^n, \Pi^i(A, k_t^*) \geq \Pi^i(A, k'_t), \quad (19)$$

where  $k_t^*$  is the process defined by (15) and (16) and the strategies  $\left( \tilde{K}^* (.,.), \tilde{I}^* (.,.) \right)$ , and  $k'_t$  is the processes created when firm  $i$  uses the strategy  $\left( \tilde{K}^i (.,.), \tilde{I}^i (.,.) \right)$  instead of  $\left( \tilde{K}^{*i} (.,.), \tilde{I}^{*i} (.,.) \right)$ , and the other firm does not change their strategies. Furthermore, a markovian strategy for firm  $i$  is said continuous if the function  $\tilde{K}^i$  is continuous, and a continuous markov perfect equilibrium is a markov perfect equilibrium with continuous strategy.

To my knowledge, this is a new way of modeling markovian strategy. In this context, it permits to properly define the best responses of the firms. In the next section, proposition 2 verifies that the best responses when firms can jump are the same than when firms can not jump. In addition, this definition permits to characterize the markov perfect equilibrium when I assume that the strategies are continuous functions of the state variable. Theorem 2 shows that the result of the one-shot game is preserved in this dynamic framework.

### 3.2 Characterization of the continuous markov equilibrium

In this subsection, I characterize the continuous markovian equilibria. I start by introducing technical assumptions. H2 is needed to prove proposition 2 (in order to use Ito's Lemma, to inverse the Ito's Lemma results and to apply theorem 1). H3 is classic to ensure the existence of a strong solution to (9). H4 ensures the existence of the stochastic integral determining the profit of the firms.



**H2:** For each  $i = 1, \dots, n$ ,  $c_i(\cdot)$  is a four times differentiable positive function such that  $c'_i \geq 0$ ,  $c''_i \geq 0$ , and for all  $A \in \mathbb{R}$ ,  $P(\cdot, \cdot)$  is also four times differentiable positive and strictly concave function in each variable. Furthermore, for all  $i = 1, \dots, n$ ,  $P''(A, \bar{q})q_i < -c''_i(q_i)$ .

**H3:**  $\beta(A)$  and  $\sigma(A)$  are continuous functions, and verify the Lipschitz conditions.

**H4:** There exists a function  $G : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , such that,  $\forall (A, x) \in \mathbb{R} \times \mathbb{R}_+$ ,  $xP(A, x) < G(A)$ , and  $\int_0^{+\infty} e^{-rt} G(A_t) dt < +\infty$ .

These assumptions allow to state proposition 2, which gives the form of the best response in the markovian state-control form of the game.

**Proposition 2:** Assume **H2**, **H3**, **H4** and **C1**. Let  $i \in \{1, \dots, n\}$ . In the markovian state-control form of the game, assume that for all  $j \neq i$ , the strategy of firm  $j$   $\tilde{K}^j(\cdot, \cdot)$  is a continuous function of the state variable. Then there exists a continuous decreasing function  $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that the best response for firm  $i$ ,  $(\tilde{K}^i_{BR}(\cdot, \cdot), \tilde{I}^i_{BR}(\cdot, \cdot))$  verifies:

$$\tilde{K}^i_{BR}(A, k) = \left\{ \begin{array}{ll} \phi(p^-) & \text{if } k^i > \phi(p^-) \\ k^i & \text{if } k^i \in [\phi(p^+), \phi(p^-)] \\ \phi(p^+) & \text{if } k^i < \phi(p^+) \end{array} \right\}, \quad (20)$$

and  $\tilde{I}^i_{BR}(A, k) = 0$  if  $k^i \in ]\phi(p^+), \phi(p^-)[$ . The best response of firm  $i$  is continuous in the state variable. ■

This proposition shows that the optimal capacity of firm  $i$  can jump: if at some time  $t$ ,  $k_t^i$  is strictly smaller than  $\phi(p^+)$ , then the firm has interest in investing instantly up to  $\phi(p^+)$ . In this case the investment in period  $t$  is infinite, so the markovian state-control form gives the same result as the regular form. However it also allows to go a step further and to characterize the equilibria, as presented in theorem 2. In fact, at each time, everything happens as in the one-shot game presented in the last section (with of course some modification of the no-move zone  $H$  in order to take into account the future profit). Firms always want to invest or disinvest forthwith in order to reach the no-move zone. As long as they are in the no-move zone, no firms change its capacity.

The value of the zone  $H$  depends of the expected profit of the firms. When firms are inside the no-move zone, the evolution of firms profits only depends on the evolution of

prices. The expected price,  $v$ , should be solution to the differential equation:

$$\forall x > 0, \quad rv(A, x) = P(A, x) + \beta(A) \frac{\partial v}{\partial A}(A, x) + \frac{\sigma^2(A)}{2} \frac{\partial^2 v}{(\partial A)^2}(A, x). \quad (21)$$

The no-move zone is then defined by:

$$H_v(A) = \left\{ k \in \mathbb{R}_+^n \mid \forall i \in \{1, \dots, n\} : v(A, \bar{k}) + \frac{\partial v}{\partial k}(A, \bar{k}) k^i - \frac{1}{r} c'_i(k^i) \in [p^-, p^+] \right\}. \quad (22)$$

When a firm reach the boundaries of the no-move zone, the marginal profit equals the price of investment (or disinvestment). This equation is the result of the optimization of the firm, and I can derive this equation with respect to the level of demand. This gives a set of condition which should be verified by the profit functions.

Assume that  $k \in H_v(A)$ , and that

$$\inf_{i \in \{1, \dots, n\}} \left\{ v(A, \bar{k}) + \frac{\partial v}{\partial k}(A, \bar{k}) k^i - \frac{1}{r} c'_i(k^i) \right\} = p^-, \quad (23)$$

or

$$\sup_{i \in \{1, \dots, n\}} \left\{ v(A, \bar{k}) + \frac{\partial v}{\partial k}(A, \bar{k}) k^i - \frac{1}{r} c'_i(k^i) \right\} = p^+. \quad (24)$$

Then, for each  $i$  which minimizes (23) or maximizes (24), the equation

$$\frac{\partial v}{\partial A}(A, \bar{k}) + k^i \frac{\partial^2 v}{\partial A \partial k}(A, \bar{k}) = 0 \quad (25)$$

should be valid. In the theory of real option, these equations are known as smooth-pasting and value-matching condition. (23) is associated to the option to invest and (24) to the option to disinvest. The conditions are not necessarily active for all the firms. Indeed, the value of waiting (i.e. the profit maid by a firm if no one invest nor disinvest) depends on the capacity holds by the firm and on its production cost.

Observe that equations (21), (23), (24) and (25) do not always define an existing or a unique solution. However, when the solution is unique, theorem 2 implies that the continuous markovian equilibrium is unique. Furthermore, each continuous markov perfect equilibrium is characterized by theorem 2.

**Theorem 2:** Assume **H2**, **H3**, **H4** and **C1**. Then, for all continuous markov perfect equilibrium  $(\tilde{K}^*(\cdot, \cdot), \tilde{I}^*(\cdot, \cdot))$ , there exists a solution  $v$  to the equations (21), (23), (24) and

(25), such that the equilibrium verifies that, for all  $(A, k) \in \mathbb{R} \times \mathbb{R}_+^n$ ,  $i \in \{1, \dots, n\}$ :

$$\tilde{K}^*(A, k) = \arg \min_{K \in H_v(A)} \sum_{i=1}^n C(K^i, k^i), \quad (26)$$

and  $\tilde{I}^*(A, k) = 0$  for all  $k \in \hat{H}_v(A)$ .

Condition (26) is equivalent to the distance condition,

$$\tilde{K}^*(A, k) = \arg \min_{K \in H_v(A)} \sum_{i=1}^n |K^i - k^i|. \quad (27)$$

Furthermore, for each solution  $v$ , the strategies defined by (26) form a continuous markovian equilibrium. ■

Theorem 2 is the analog of theorem 1, but in a continuous time setting. It characterizes the continuous markovian equilibria. At each time, firms invest (or disinvest) in order to join the no-move zone at the smallest possible cost for the industry. The efficiency of investment observed in the one-shot model is thus still valid in a dynamics and uncertain framework.

This theorem focuses on the continuous markovian equilibrium. Nevertheless, one can ask about other markovian equilibria. Indeed, there is a priori no reason that collusion can not be sustained by markovian strategies in a differential game. In a companion paper on the same investment game, [Fagart \(2013\)](#), I exhibit a markovian equilibrium with tit-for-tat strategy implementing the monopoly profit.<sup>8</sup>

## 4 Conclusion

In this work, I characterize the continuous markovian equilibria of a model of partially irreversible investment in capacity under uncertainty and Cournot competition. There exists

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<sup>8</sup>Such collusive equilibria arise because we assume that the strategy of the players rests on the state variable. In the classic modeling of differential games, the strategies of the players leads on the derivative of the state variable, which imposes that the state variable will evolve continuously with time. As shown by theorem 2 in our case, this continuity assumption provides the finding of collusive equilibria when the strategy of the player rests on the derivative of the state variable. If the finding of collusive equilibria is an argument in favor of our modeling, it also asks why the monopoly profit can not be implemented by continuous strategies.

an area in the space of firms capacities, the no-move zone, such that firms invest or disinvest in order to join this area as soon as possible, and keep their capacity constant when they are inside this area. The existence of this area is due to the irreversibility of investment, and when investment is perfectly reversible, the no-move zone is reduced to a single point, as in usual Cournot competition. When firms have a similar profit function, but different initial capacities, their asymmetry can be preserved, due to the irreversibility of investment.

At the equilibrium, firms install the capacities vector of the no-move zone which minimizes the cost of investment of the all industry. The intuition on this result is that the no-move is the area where other firms' actions do not impact the optimal action of a firm, so the efficiency of the equilibrium comes from the absence of reaction to competitor action inside the no-move zone.

## 5 Appendix

### Proof of theorem 1:

First, note that the profit function (3) verifies the hypothesis of theorem 3 of [Novshek \(1985\)](#), so there exists an unique Nash equilibrium. Let

$$\hat{K} = \arg \min_{K \in H} \sum_{i=1}^n |K^i - k^i|. \quad (28)$$

Then, if I show that  $\hat{K}$  is an equilibrium, it is unique. As (28) has always a solution, this will prove theorem 1.

The first step of this proof introduces some notation. For any  $i \in \{1, \dots, n\}$ , I will assume that  $\hat{K}^i$  is not the best response to  $\left(\hat{K}^j\right)_{j \neq i}$ . The second step assume that  $\hat{K}^i > k^i$  and introduce a vector of capacity  $\check{K}$ , which belongs to  $H$  and is closer to  $k$  than  $\hat{K}$ . This contradicts (28) and implies that  $\hat{K}^i$  is a best response when  $\hat{K}^i > k^i$ . The third step presents the same reasoning when  $\hat{K}^i < k^i$  and concludes. The last step shows the equivalence between the two conditions (7) and (8).

**Step 1:** Some definition.

Let  $x_i = G_i^-(x), G_i^+(x)$  be the implicit functions (these functions are well defined and  $C^1$  by assumption H1) defined by:

$$P(x) + P'(x)x_i - c'_i(x_i) = p^-, P(x) + P'(x)x_i - c'_i(x_i) = p^+. \quad (29)$$

Observe that the best response of firm  $i$  verifies:

$$K^i = \left\{ \begin{array}{l} G_i^+(\bar{K}) \text{ if } k^i < G_i^+(\bar{K}) \\ k^i \text{ if } G_i^+(\bar{K}) \leq k^i \leq G_i^-(\bar{K}) \\ G_i^-(\bar{K}) \text{ if } k^i > G_i^-(\bar{K}) \end{array} \right\}. \quad (30)$$

Thus,  $K$  is an equilibrium if and only if (30) holds for all  $i$ . The no-move zone can be rewritten as

$$H = \{K \in \mathbb{R}_+^n \mid \forall i \in \{1, \dots, n\}, K^i \in [G_i^+(\bar{K}), G_i^-(\bar{K})]\}. \quad (31)$$

**Step 2:**  $\hat{K}^i$  is firm's  $i$  best response when  $\hat{K}^i > k^i$ .

This step is subdivided in three intermediate stages. The first stage constructs the candidate point for the contradiction,  $\check{K}$ . The second stage shows that  $\check{K} \in H$  if and only if (37) is valid. The third stage verifies that inequalities of (37) are valid.

· First, remark that, as  $\hat{K}^i > k^i$ ,  $\hat{K}^i$  is firm's  $i$  best response if and only if  $\hat{K}^i = G_i^+(\sum_{h=1}^n \hat{K}^h)$ . Assume that  $\hat{K}^i$  is not firm's  $i$  best response. Then, as  $\hat{K} \in H$ ,  $\hat{K}^i > G_i^+(\sum_{h=1}^n \hat{K}^h)$ .

Let  $\Omega = \{j : \hat{K}^j = G_j^+(\sum_{h=1}^n \hat{K}^h)\}$ ,  $z = \text{card}(\Omega)$  and  $\check{K}$  defined by:

$$\check{K}^j = \left\{ \begin{array}{l} \hat{K}^i - \varepsilon \text{ if } j = i \\ \hat{K}^j + \frac{\varepsilon}{z+1} \text{ if } j \in \Omega \\ \hat{K}^j \text{ elsewhere} \end{array} \right\}. \quad (32)$$

For  $\varepsilon$  small enough to have  $\check{K}^i > k^i$ , observe that:

$$\sum_{h=1}^n |\check{K}^h - k^h| = \sum_{h=1}^n |\hat{K}^h - k^h| - \frac{\varepsilon}{z+1}. \quad (33)$$

To contradict the fact that  $\hat{K}^i$  is not firm's  $i$  best response, I just need to show that  $\hat{K}^i > G_i^+(\sum_{h=1}^n \hat{K}^h)$  implies that  $\check{K} \in H$ .

· Observe that

$$\sum_{h=1}^n \check{K}^h = \sum_{h=1}^n \hat{K}^h - \frac{\varepsilon}{z+1}. \quad (34)$$

By hypothesis H1,  $G_j^u(x) = -\frac{P'(x)+P''(x)x_j}{P'(x)-c_i''(x_j)} < 0$  and, as  $P''(x)x_j \geq -c_j''(x_j)$ ,  $G_j^u > -1$ ,  $u \in \{+, -\}$ . By Taylor, there exists  $\lambda_j^u \in ]0, 1[$  such that

$$G_j^u \left( \sum_{h=1}^n \check{K}^h \right) = G_j^u \left( \sum_{h=1}^n \hat{K}^h \right) + \lambda_j^u \frac{\varepsilon}{z+1}. \quad (35)$$

Furthermore, when  $\varepsilon$  is small enough,

$$G_j^- \left( \sum_{h=1}^n \check{K}^h \right) > G_j^+ \left( \sum_{h=1}^n \hat{K}^h \right) + \frac{\varepsilon}{z+1}. \quad (36)$$

Thus,  $\check{K}$  belongs to  $H$  if and only if

$$G_j^+ \left( \sum_{h=1}^n \check{K}^h \right) \leq \check{K}^j \leq G_j^- \left( \sum_{h=1}^n \check{K}^h \right), \quad (37)$$

for all  $j \in \{1, \dots, n\}$ .

· When  $j \notin \Omega$ , I have  $\check{K}^j \leq \hat{K}^j \leq G_j^- \left( \sum_{h=1}^n \hat{K}^h \right) < G_j^- \left( \sum_{h=1}^n \check{K}^h \right)$ . If  $j \in \Omega$ , from (36) and (35),  $\check{K}^j = \hat{K}^j + \frac{\varepsilon}{z+1} = G_j^+ \left( \sum_{h=1}^n \hat{K}^h \right) + \frac{\varepsilon}{z+1} \leq G_j^- \left( \sum_{h=1}^n \hat{K}^h \right) \leq G_j^- \left( \sum_{h=1}^n \check{K}^h \right)$ . Thus  $\check{K}^j \leq G_j^- \left( \sum_{h=1}^n \check{K}^h \right)$  for all  $j$ .

When  $j \in \Omega$ ,

$$\check{K}^j = \hat{K}^j + \frac{\varepsilon}{z+1} = G_j^+ \left( \sum_{h=1}^n \hat{K}^h \right) + \frac{\varepsilon}{z+1} = G_j^+ \left( \sum_{h=1}^n \check{K}^h \right) + \frac{\varepsilon}{z+1} - \lambda_j^u \frac{\varepsilon}{z+1} \geq G_j^+ \left( \sum_{h=1}^n \check{K}^h \right). \quad (38)$$

The first equality comes from (32), the second one from the definition of  $\Omega$ , and the third one from (35). The inequality holds as  $\lambda_j^u \in ]0, 1[$ . Finally, when  $j \notin \Omega$ , by construction of  $\Omega$ ,  $\hat{K}^j > G_j^+ \left( \sum_{h=1}^n \hat{K}^h \right)$ , so, for  $\varepsilon$  small enough, I have:

$$\hat{K}^j \geq G_j^+ \left( \sum_{h=1}^n \hat{K}^h \right) + \lambda_j^+ \frac{\varepsilon}{z+1} + \varepsilon.$$

Thus, by (35),  $\check{K}^j \geq G_j^+ \left( \sum_{h=1}^n \check{K}^h \right)$ , and  $\check{K} \in H$ .

**Step 3:**  $\hat{K}^i$  is firm's  $i$  best response when  $\hat{K}^i < k^i$ .

As step 2, this step is subdivide in three intermediate stage.

· When  $\hat{K}^i < k^i$ ,  $\hat{K}^i$  is firm's  $i$  best response if and only if  $\hat{K}^i = G_i^-(\sum_{h=1}^n \hat{K}^h)$ .

Assume that  $\hat{K}^i$  is not firm's  $i$  best response. Then, as  $\hat{K} \in H$ ,  $\hat{K}^i < G_i^-(\sum_{h=1}^n \hat{K}^h)$ .

Let  $\Omega = \left\{ j : \hat{K}^j = G_j^-\left(\sum_{h=1}^n \hat{K}^h\right) \right\}$ ,  $z = \text{card}(\Omega)$  and  $\bar{K}$  defined by:

$$\check{K}^j = \begin{cases} \hat{K}^i + \varepsilon & \text{if } j = i \\ \hat{K}^j - \frac{\varepsilon}{z+1} & \text{if } j \in \Omega \\ \hat{K}^j & \text{elsewhere} \end{cases}. \quad (39)$$

Observe that, for small  $\varepsilon$ :

$$\sum_{h=1}^n \left| \check{K}^h - k^h \right| = \sum_{h=1}^n \left| \hat{K}^h - k^h \right| - \frac{\varepsilon}{z+1}, \quad (40)$$

and

$$\sum_{h=1}^n \check{K}^h = \sum_{h=1}^n \hat{K}^h + \frac{\varepsilon}{z+1}. \quad (41)$$

· Now, (35) and (36) became, for  $\lambda_j^u \in ]0, 1[$ :

$$G_j^u \left( \sum_{h=1}^n \check{K}^h \right) = G_j^u \left( \sum_{h=1}^n \hat{K}^h \right) - \lambda_j^u \frac{\varepsilon}{z+1}. \quad (42)$$

Furthermore, when  $\varepsilon$  is small enough,

$$G_j^- \left( \sum_{h=1}^n \check{K}^h \right) > G_j^+ \left( \sum_{h=1}^n \hat{K}^h \right) + \frac{\varepsilon}{z+1}. \quad (43)$$

· When  $j \notin \Omega$ , we have  $\check{K}^j \geq \hat{K}^j \geq G_j^-\left(\sum_{h=1}^n \hat{K}^h\right) > G_j^-\left(\sum_{h=1}^n \check{K}^h\right)$ . If  $j \in \Omega$ , from (43) and (42),  $\check{K}^j = \hat{K}^j - \frac{\varepsilon}{z+1} = G_j^-\left(\sum_{h=1}^n \hat{K}^h\right) - \frac{\varepsilon}{z+1} \geq G_j^+\left(\sum_{h=1}^n \hat{K}^h\right) \geq G_j^-\left(\sum_{h=1}^n \check{K}^h\right)$ . Thus  $\check{K}^j \leq G_j^-\left(\sum_{h=1}^n \check{K}^h\right)$ .

When  $j \in \Omega$ , (39) and (42) gives:

$$\check{K}^j = \hat{K}^j - \frac{\varepsilon}{z+1} \leq G_j^-\left(\sum_{h=1}^n \hat{K}^h\right) - \frac{\varepsilon}{z+1} = G_j^-\left(\sum_{h=1}^n \check{K}^h\right) - \frac{\varepsilon}{z+1} + \lambda_j^u \frac{\varepsilon}{z+1} \leq G_j^-\left(\sum_{h=1}^n \check{K}^h\right),$$

whereas, when  $j \notin \Omega$ , for  $\varepsilon$  small enough, we have:

$$\hat{K}^j \leq G_j^-\left(\sum_{h=1}^n \hat{K}^h\right) - \lambda_j^+ \frac{\varepsilon}{z+1} - \varepsilon,$$

and thus,  $\check{K}^j \leq G_j^- \left( \sum_{h=1}^n \check{K}^h \right)$ . Therefore  $\check{K} \in H$ .

Observe that if  $\hat{K}^i = k^i$ ,  $k^i \in H$  as  $\hat{K}^i \in H$ , and (30) and (31) implies that  $\hat{K}^i$  is firm  $i$  best response. This prove that the  $\hat{K}$  is an equilibrium.

**Step 3:** Equivalence between the two equilibrium characterization (7) and (8).

To show the equivalence between (7) and (8), it suffices to remark that equations (33) and (40) become respectively

$$C(\check{K}, k) = C(\hat{K}, k) - p^+ \frac{1}{z+1} \varepsilon \quad (44)$$

and

$$C(\check{K}, k) = C(\hat{K}, k) - p^- \frac{1}{z+1} \varepsilon \quad (45)$$

with the investment cost function. Thus, the same reasoning applies. ■

In the following proofs, I put the time in index when I consider a variable as a stochastic process and nothing in index when I consider the variable as a constant. Thereby,  $K^i = \tilde{K}^i(A, k)$  is the capacity implemented by firm  $i$  for a level of demand  $A$  and a vector of installed capacity  $k$ , whereas  $K_t^i = \tilde{K}^i(A_t, k_t)$  is the stochastic process implemented during a period of time.

### Proof of proposition 2:

This proof is made in three step. The properties of the profit function exhibited in step 1 are used in step 2 to find the optimal control when firm  $i$  does not jump. Step 3 describes the optimal jump and concludes.

#### Step 1: Properties on $\Pi^i$ .

Assume that the optimal control,  $\tilde{K}^{*i}(\cdot)$  is a continuous function. Then, as  $\tilde{K}^j(\cdot)$  is also continuous, the path  $k_t$  is continuous (except in 0). Indeed, when  $h \rightarrow 0$ ,

$$k_{t+h} = \tilde{K}(k_{t+h}) \rightarrow \tilde{K}(k_t) = k_t. \quad (46)$$

Thus, if there is a jump, it is done only at time 0, and after that the firm uses it continuous control,  $I^i$ , to adapt continuously the state variable  $k_t^i$ . Let  $J$  be the set of initial conditions such that firm  $i$  does not jump,

$$J = \left\{ (A, k) \mid \tilde{K}^{*i}(A, k) = k^i \right\}. \quad (47)$$



By assumption C1, this space is non empty. If  $(A, k)$  belongs to  $J$ , Bellmans verification theorem applies as the state space variable evolves continuously, and the objectif function,  $\Pi^i$ , is  $C^2$ .

Furthermore,  $\Pi^i$  is an increasing function of  $k^i$ . Indeed, assume that  $k$  and  $k'$  are two initial conditions such that  $k^i > k'^i$  and  $k^j = k'^j$  if  $j \neq i$ . Then, let  $k_t'^{*i}$  be the optimal path induced by  $k'$ . If the initial condition are  $(A, k)$ , firm  $i$  can always imitate the optimal strategy coming from  $(A, k')$ , investing according to  $k'^i$  instantly and following  $k_t'^{*i}$  after. In this case,

$$\Pi^i(A, k) \geq \Pi^i(A, k'^i) + p^- (k^i - k'^i). \quad (48)$$

Thus,  $\Pi^i$  is an increasing function.

**Step 2:** Optimal strategy when the firm does not jump.

If the initial point  $(A, k)$  belongs to  $J$ , Bellman theorem applies and:

$$r\Pi^i(A, k) = \max_{I^i} \left\{ \pi^i(A, k) - p^+ (I^i)_+ - p^- (I^i)_- + I^i \frac{\partial \Pi^i}{\partial K^i} + D\Pi^i \right\}, \quad (49)$$

where  $D\Pi^i$  does not depend of the control  $I^i$ . If  $\frac{\partial \Pi^i}{\partial K^i} \in [p^-, p^+]$ , then the optimal control is  $I^i = 0$ . If  $\frac{\partial \Pi^i}{\partial K^i} \notin [p^-, p^+]$ , then there is no optimal control available in  $\mathbb{R}$ , and in that case,  $(A, k) \notin J$ , i.e. firm  $i$  wishes to jump.

**Step 3:** Optimal strategy when the firm does jump.

Assume now that the optimal strategy of firm  $i$  is to jump at the initial point  $(A, k)$ . Let  $K^i$  be the level of the jump and  $\hat{k}^i$  be the vector of capacity at the instant of the jump of firm  $i$ :

$$\check{k}^i = K^i, \check{k}^j = \tilde{K}^j(k) \text{ for } j \neq i. \quad (50)$$

Then, the profit of firm  $i$  is equal to:

$$\Pi^i(A, k) = \Pi^i(A, \hat{k}) - C(\hat{k}^i, k_0^i). \quad (51)$$

After the jump,  $(A, \hat{k})$  belongs to  $J$  and thus  $\Pi^i$  is derivable in  $(A, \hat{k})$ . As  $\Pi^i$  is increasing, the solution to (51) is unique and given by:

$$K^i = \left\{ \begin{array}{ll} \phi(p^-) & \text{if } k^i > \phi(p^-) \\ k^i & \text{if } k^i \in [\phi(p^+), \phi(p^-)] \\ \phi(p^+) & \text{if } k^i < \phi(p^+) \end{array} \right\}, \quad (52)$$

where  $\phi$  is the implicit function defined by

$$\frac{\partial \Pi^i}{\partial k^i}(A, k_\phi) = x, \quad (53)$$

for  $k_\phi^i = \phi(x)$  and  $k_\phi^j = k^j$ . In that case,  $J = [\phi(p^+), \phi(p^-)]$ , and the strategy is  $I^i = 0$  except when  $k^i = \{\phi(p^+), \phi(p^-)\}$ .

Remark that if  $\Pi^i$  is  $C^2$  for all its variables,  $D\Pi^i$  is given by

$$D\Pi^i = \beta(A) \frac{\partial \Pi^i}{\partial A} + \frac{\sigma^2(A)}{2} \frac{\partial^2 \Pi^i}{(\partial A)^2} + \sum_{\substack{j=1 \\ j \neq i}}^n \tilde{I}^j \frac{\partial \Pi^i}{\partial k^j} + \frac{\sigma^2(A)}{2} \sum_{\substack{j,h=1 \\ j,h \neq i}}^n \tilde{I}^j \tilde{I}^h \frac{\partial^2 \Pi^i}{\partial k^j \partial k^h}. \quad (54)$$

■

## Proof of Theorem 2:

The idea of proof is to characterizes the profit function inside the no-move zone (where the optimal firms strategies are to keep their capacities constant), and to apply theorem 1 in order to characterize the equilibria.

Let  $(\tilde{K}^*(.,.), \tilde{I}^*(.,.))$  be a continuous markov perfect equilibrium. For all  $A \in \mathbb{R}$ , let  $H(A)$  be the no-move zone defined by,

$$H(A) = \left\{ k \in \mathbb{R}_+^n \mid \tilde{K}^*(A, k) = k \right\}. \quad (55)$$

By assumption C1, this no-move zone is not empty. Proposition 2 states that when the installed capacity  $k$  does not belong to  $H$ , firms wish to invest or disinvest in order to install a vector of capacity which belongs to  $H$ . The first two steps of the proof characterize the profit of the firm inside the no-move zone. The first step gives the dynamic equation verified by the profit functions. The second step presents the smooth-pasting and value matching conditions. The third step establishes that the application requirements for theorem 1 are valid. Step 4 applies theorem 4, gives the form of the no-move zone and characterizes the equilibrium. Finally, step 5 shows that every strategies defined by equations (21), (23), (24), (25) and (26) is an equilibrium.

**Step 1:** The profit functions as a solution to a unique differential equation, (58).

Assume that  $k \in H$ . In this case, the best strategy of each firm is to maintain its capacity constant and thus the firms profit depends only on the evolution of the uncertainty. Therefore, for all  $i \in \{1, \dots, n\}$ ,

$$D\Pi^i = \beta(A) \frac{\partial \Pi^i}{\partial A}(A, k) + \frac{\sigma^2(A)}{2} \frac{\partial^2 \Pi^i}{(\partial A)^2}(A, k). \quad (56)$$

Furthermore, when  $k$  belongs to  $H$ , the equilibrium strategies are  $\tilde{K}^*(A, k) = k$ , and thus, by (49):

$$r\Pi^i(A, k) = P(A, \bar{k}) k^i - c_i(k^i) + \beta(A) \frac{\partial \Pi^i}{\partial A}(A, k) + \frac{\sigma^2(A)}{2} \frac{\partial^2 \Pi^i}{(\partial A)^2}(A, k). \quad (57)$$

Cauchy theorem shows that the space of solutions to a second degree differential equation is a two-dimensional vector space. The set of solution to (57) is thus a priori two- $n$ -dimensional. However, (57) can be reduced to a unique equation, and the set of solutions to a two-dimensional space. Indeed, if  $v(., .)$  is a solution to

$$rv(A, x) = P(A, x) + \beta(A) \frac{\partial v}{\partial A}(A, x) + \frac{\sigma^2(A)}{2} \frac{\partial^2 v}{(\partial A)^2}(A, x), \quad (58)$$

the profits defined by

$$\forall i \in \{1, \dots, n\}, \Pi^i(A, k) = v(A, \bar{k}) k^i - \frac{1}{r} c_i(k^i), \quad (59)$$

is solution to the equations (57). Reciprocally, a solution to (57) defines a solution to (58).

**Step 2:** Smooth pasting and value-matching condition.

Let  $k \in H(A)$ . Let  $A^+$  be the level of demand such that one firm invests if the demand increases over this level. Then, if firm  $i$  is a firm which invests just after  $A^+$ , the profit function of firm  $i$  should verify,

$$\frac{\partial \Pi^i}{\partial k^i}(A^+, k) = p^+. \quad (60)$$

If firm  $i$  does not invest just after  $A^+$ ,

$$\frac{\partial \Pi^i}{\partial k^i}(A^+, k) < p^+. \quad (61)$$

The smooth pasting condition is thus

$$\sup_{i \in \{1, \dots, n\}} \frac{\partial \Pi^i}{\partial k^i}(A^+, k) = p^+, \quad (62)$$

which can be written as:

$$\sup_{i \in \{1, \dots, n\}} \left\{ v(A^+, \bar{k}) + \frac{\partial v}{\partial k}(A^+, \bar{k}) k^i - \frac{1}{r} c'_i(k^i) \right\} = p^+. \quad (63)$$

The differentiation of (60) gives the optimal matching condition:

$$\frac{\partial^2 \Pi^i}{\partial A \partial k^i}(A^+, k) = 0, \quad (64)$$

for all  $i$  which maximizes (62), which can be written

$$\frac{\partial v}{\partial A}(A^+, \bar{k}) + \frac{\partial^2 v}{\partial A \partial k^i}(A^+, \bar{k}) k^i = 0. \quad (65)$$

The same reasoning applies for disinvesting when the level of demand decreases, and in that case the smooth-pasting and value matching conditions are:

$$\inf_{i \in \{1, \dots, n\}} \left\{ v(A^-, \bar{k}) + \frac{\partial v}{\partial k}(A^-, \bar{k}) k^i - \frac{1}{r} c'_i(k^i) \right\} = p^-, \quad (66)$$

and

$$\frac{\partial v}{\partial A}(A^-, \bar{k}) + \frac{\partial^2 v}{\partial A \partial k^i}(A^-, \bar{k}) k^i = 0. \quad (67)$$

Equations (63), (65), (66) and (67) characterizes the solution of the differential equation (58).

At this point of the proof, we have seen that the profit function of firm  $i$ ,  $i \in \{1, \dots, n\}$ , can be rewritten:

$$\Pi^i(A, k) = \max_{K^i | K \in H(A)} \left\{ v(A, \bar{K}) K^i - \frac{1}{r} c_i(K^i) - C(K^i, k^i) \right\}, \quad (68)$$

where  $v$  is a solution to the differential equation (58), which verifies (63), (65), (66) and (67).

### Step 3: Verification of assumption H1.

The principal difficulty of that step is that (58) defines  $v$  only on  $H$ . So I will use a general solution to (58),  $w$ , and show that assumption H1 is valid for this function, before to bring me to  $v$ .

Let  $w(A, x)$  be a solution to (58) for all  $(A, x) \in \mathbb{R}_+^2$ . Using Itô Lemma on the function  $e^{-rt}w(A_t, \bar{k})$  gives:

$$\begin{aligned} d(e^{-rt}w(A_t, \bar{k})) &= \left( \beta(A_t) \frac{\partial w}{\partial A}(A_t, \bar{k}) + \frac{\sigma^2(A_t)}{2} \frac{\partial^2 w}{(\partial A)^2}(A_t, \bar{k}) - rw(A_t, \bar{k}) \right) e^{-rt} dt \\ &\quad + \sigma(A_t) e^{-rt} \frac{\partial w}{\partial A}(A_t, \bar{k}) dW_t. \end{aligned}$$

By (58), this equation becomes

$$d(e^{-rt}w(A_t, \bar{k})) = -P(A_t, \bar{k})e^{-rt}dt + \sigma(A_t)e^{-rt}\frac{\partial w}{\partial A}(A_t, \bar{k})dW_t, \quad (69)$$

which can be rewritten, for  $h > 0$ ,

$$\forall A_0 \in \mathbb{R}_+, w(A_0, \bar{k}) - e^{-rh}w(A_h, \bar{k}) = \int_0^h P(A_t, \bar{k})e^{-rt}dt - \int_0^h \sigma(A_t)e^{-rt}\frac{\partial w}{\partial A}(A_t, \bar{k})dW_t \quad (70)$$

and thus

$$w(A_0, \bar{k}) - e^{-rh}E[w(A_h, \bar{k})|A_0] = E\left[\int_0^h P(A_t, \bar{k})e^{-rt}dt \mid A_0\right]. \quad (71)$$

When  $h \rightarrow +\infty$ , (71) becomes,

$$w(A_0, \bar{k}) = E\left[\int_0^{+\infty} P(A_t, \bar{k})e^{-rt}dt \mid A_0\right]. \quad (72)$$

Assumption H4 implies that this integral is well defined. Assumption H2 implies that  $v' < 0$ ,  $v'' < 0$  and:

$$w''(A, \bar{k}) = E\left[\int_0^{+\infty} e^{-rt}P''(A_t, \bar{k})dt \mid A\right] < E\left[\int_0^{+\infty} e^{-rt}\frac{c_i''(\bar{k})}{\bar{k}}dt \mid A\right] = \frac{1}{r}\frac{c_i''(\bar{k})}{\bar{k}}. \quad (73)$$

So  $w$  verifies assumption H1. Furthermore, on  $H$ ,  $w(A, \bar{k})$  is a particular solution to the differential equation (58), and, as  $v$  is also a solution to (58), there exists a function  $g(\cdot)$  such that, on  $H$ :

$$v(A, \bar{k}) = g(A) + w(A, \bar{k}), \quad (74)$$

where  $g$  is a solution to the differential equation:

$$rg = \beta(A)\frac{\partial g}{\partial A} + \frac{\sigma^2(A)}{2}\frac{\partial^2 g}{(\partial A)^2}. \quad (75)$$

Outside the zone  $H$ ,  $v$  is not necessarily equal to  $g + w$ . However, the profit functions do not depend on  $v$  outside  $H$ , and equation (68) is equivalent to:

$$\Pi^i(A, k) = \max_{K^i | K \in H} \left\{ (g(A) + w(A, \bar{k}))K^i - \frac{1}{r}c_i(K^i) - C(K^i, k^i) \right\}. \quad (76)$$

**Step 4:** Applying theorem 1 and showing the unicity of the equilibrium.

As  $g + w$  verifies assumption  $H1$ , I can apply theorem 1. This gives the characterization of the equilibrium (26), and the form of zone  $H$ :

$$H(A) = \left\{ k \in \mathbb{R}_+^n \mid \forall i \in \{1, \dots, n\} : g(A) + w(A, \bar{k}) + w'(A, \bar{k})k^i - \frac{1}{r}c'_i(k^i) \in [p^-, p^+] \right\}. \quad (77)$$

Of course, by (74), this can be rewritten as:

$$H(A) = \left\{ k \in \mathbb{R}_+^n \mid \forall i \in \{1, \dots, n\} : v(A, \bar{k}) + v'(A, \bar{k})k^i - \frac{1}{r}c'_i(k^i) \in [p^-, p^+] \right\}. \quad (78)$$

**Step 5:** Equation (21), (23), (24), (25) and (26) define an equilibrium.

Assume that there exists a solution  $v$  to the equations (21), (23), (24), and (25), and let  $\tilde{K}(\cdot, \cdot)$  be the equilibrium defined by (26) (with  $\tilde{I} = 0$ ). For all  $(A, k) \in \mathbb{R} \times \mathbb{R}_+^n$ , let  $K = \tilde{K}(A, k)$  be the vector of desired capacities. Then, for all  $i \in \{1, \dots, n\}$ , I show that  $K^i$  is the best response of firm  $i$  to  $(K^j)_{j \neq i}$ . Indeed, assume that  $(K^j)_{j \neq i}$  is fixed, then, by (26), the strategy of firm  $i$  verifies,

$$\tilde{K}^i(A, k) = \arg \max_{K^i \in H^i_v(A)} C(K^i, k^i), \quad (79)$$

where

$$H^i_v(A) = \left\{ K^i \in \mathbb{R}_+ \mid v(A, \bar{K}) + v'(A, \bar{K})K^i - \frac{1}{r}c'_j(K^i) \in [p^-, p^+] \right\}. \quad (80)$$

As  $v$  is a solution to (21),  $v$  verifies  $H1$  by step 3. Thus, for all  $x \in [p^-, p^+]$ , there exists a unique solution to the implicit equation,

$$v(A, \bar{K}) + v'(A, \bar{K})K^i - \frac{1}{r}c'_j(K^i) = x, \quad (81)$$

as a unique solution  $K^i = \phi(x)$ . In that case, the strategy of firm  $i$  is defined by

$$K^i = \left\{ \begin{array}{ll} \phi(p^-) & \text{if } k^i > \phi(p^-) \\ k^i & \text{if } k^i \in [\phi(p^+), \phi(p^-)] \\ \phi(p^+) & \text{if } k^i < \phi(p^+) \end{array} \right\}. \quad (82)$$

Furthermore, even if  $\tilde{K}(\cdot, \cdot)$  is not an equilibrium, firms does not invest nor disinvest inside the no-move zone, and the reasoning of step 1 and step 2 applies. Thus, for  $k \in H(A)$ ,

$$\frac{\partial \Pi^i}{\partial k^i} = v(A, \bar{k}) + v'(A, \bar{k})k^i - \frac{1}{r}c'_j(k^i). \quad (83)$$

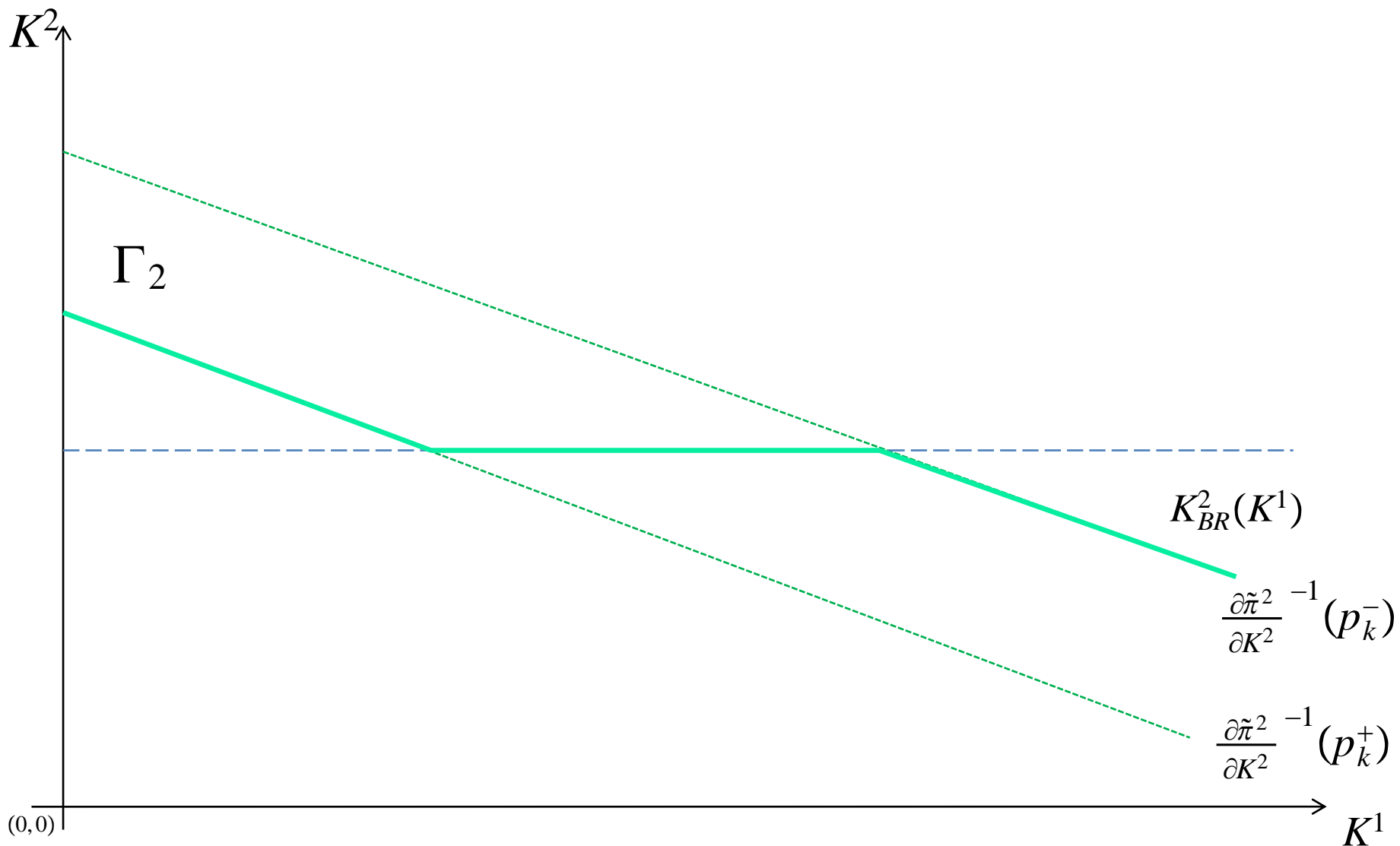
Equations (82) and (83) are equivalent to (52) and (53), so by proposition 2 the strategy of firm  $i$  is its best response to the opponents strategy, and  $\tilde{K}(.,.)$  is a markov perfect equilibrium. ■

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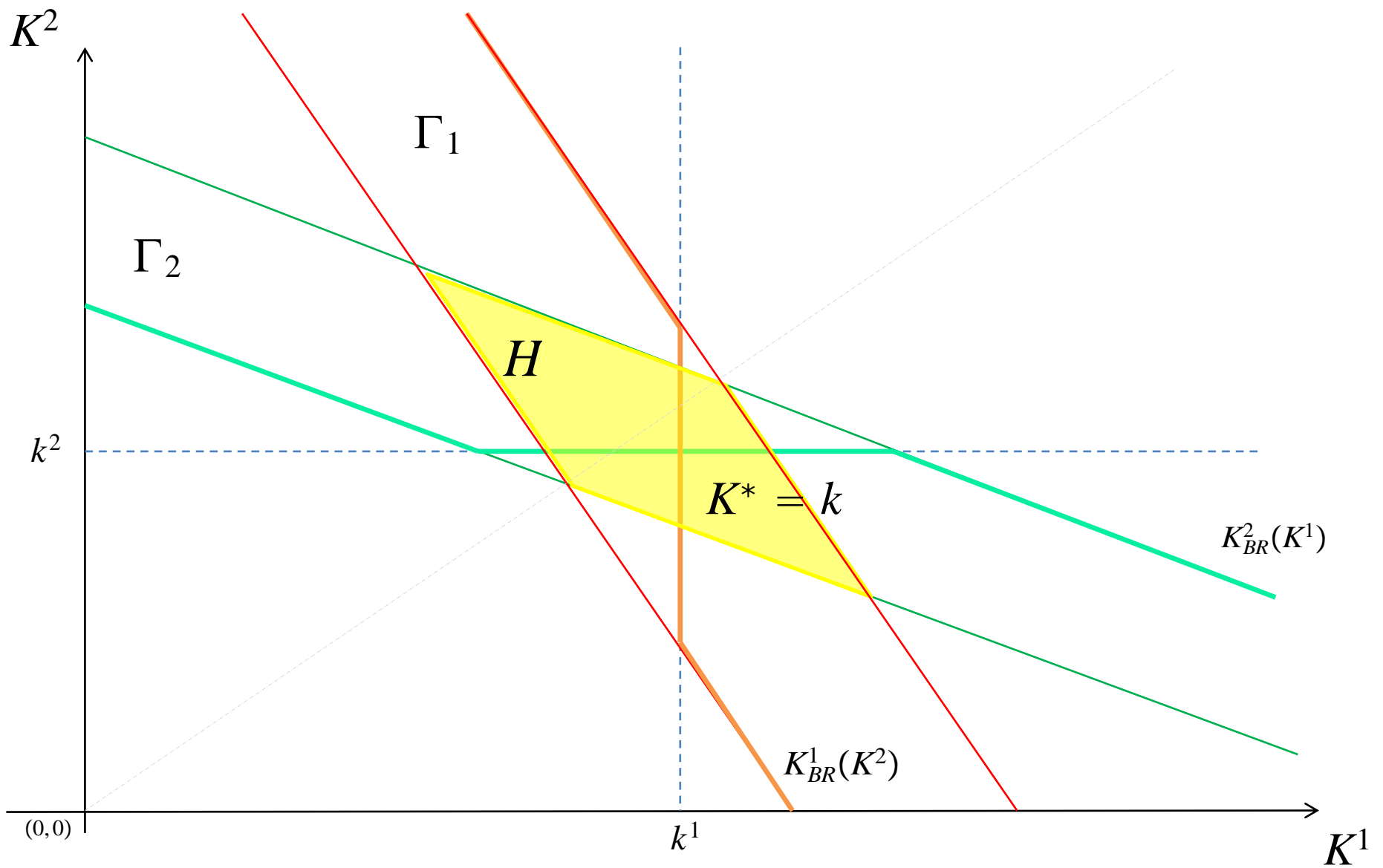
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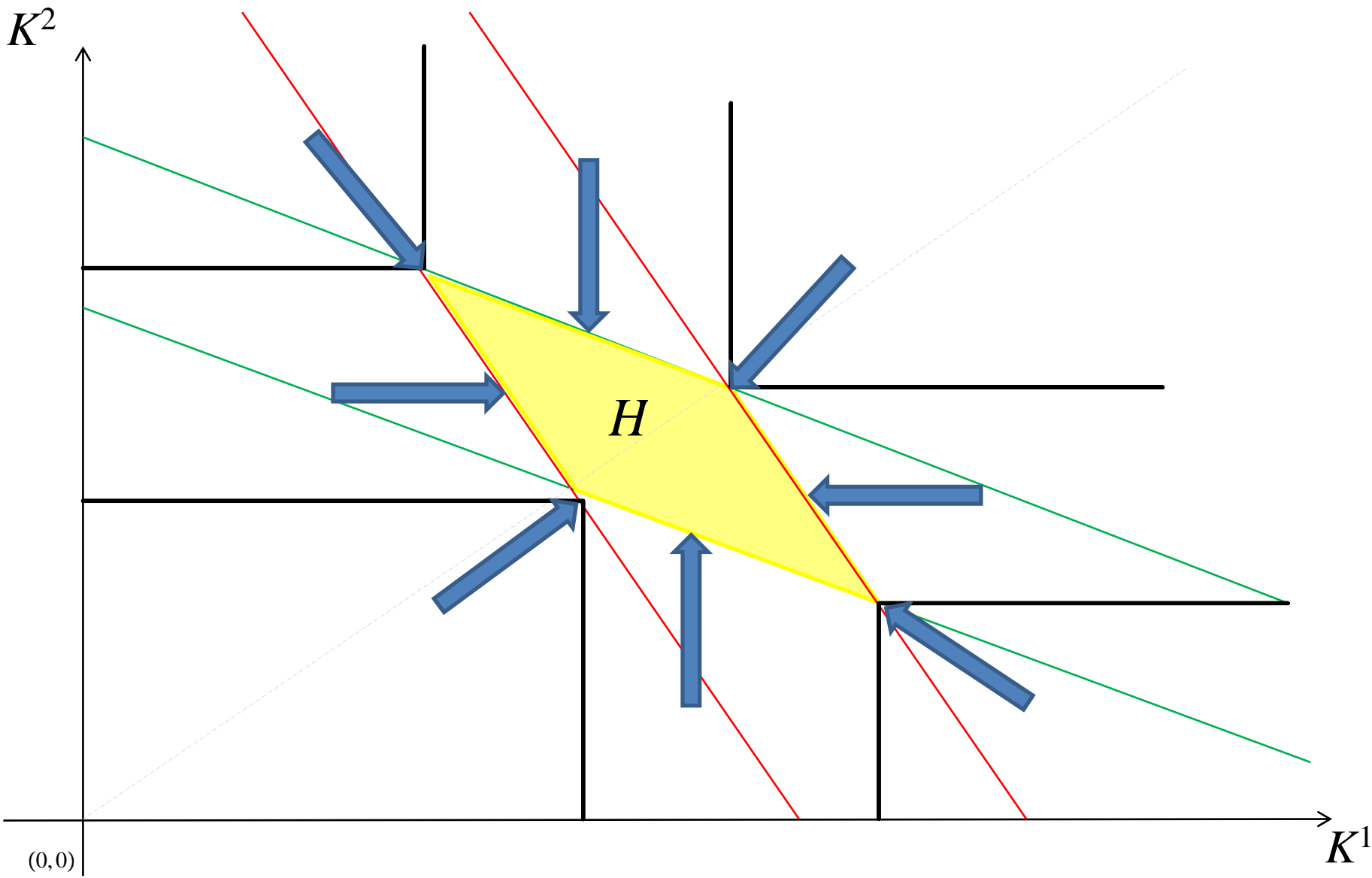
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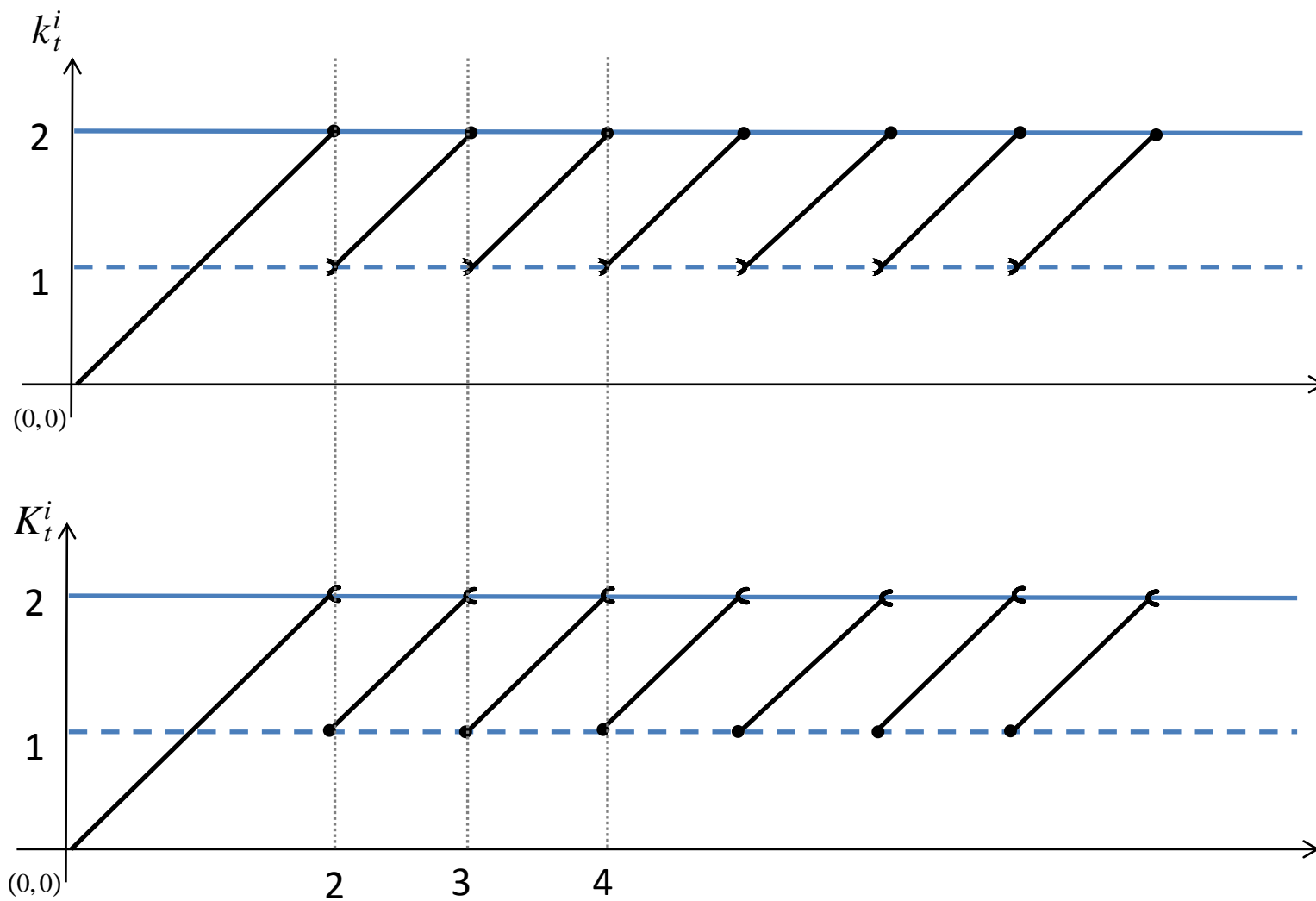
Graphic 1: best response of firm 2



Graphic 2: No-move zone and equilibrium



Graphic 3: Joining the no-move



Graphic 4: strategy with jump